Exercise 1. Let $f \in X'$. Since $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to x_∞, we have $f(x_n) \underset{n\to\infty}{\longrightarrow} f(x_\infty)$. Therefore, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|f(x_n) - f(x)| < \varepsilon$. Therefore, we get by linearity for all $m \geq N$

$$\left|f\left(\frac{1}{m}\sum_{i=1}^m x_i\right) - f(x_\infty)\right| \leq \frac{1}{m}\sum_{k=1}^m |f(x_i) - f(x_\infty)| \leq \frac{1}{m}\sum_{i=1}^N \|f\|_{X'}\left(\|x_n\|_X + \|x_\infty\|_X\right) + \frac{m-N}{m}\varepsilon.$$

Since $\{\|x_n\|_X\}_{n\in\mathbb{N}}$ is bounded (as $\{x_n\}_{n\in\mathbb{N}}$ converges weakly), we deduce that

$$\limsup_{m\to\infty}\left|f\left(\frac{1}{m}\sum_{i=1}^m x_i\right) - f(x_\infty)\right| \leq \varepsilon.$$

As the result is true for all $\varepsilon > 0$, the weak convergence follows.

Exercise 2. We have for all $1 \leq p < \infty$

$$\int_0^{2\pi} |\sin(nx)|^p dx = \sum_{k=0}^{n-1}\int_{\frac{2k\pi}{n}}^{\frac{2\pi(k+1)}{n}} |\sin(nx)|^p dx = \frac{1}{n}\sum_{k=0}^{n-1}\int_{2\pi k}^{2\pi(k+1)} |\sin(y)|^p dy = \int_0^{2\pi} |\sin(y)|^p dy > 0$$

by 2π-periodicity. Therefore, the sequence never converges to 0 strongly. The weak converge follows from the Riemann-Lebesgue theorem. Assuming without loss of generality that $\varphi \in C_c^\infty(]0, 2\pi[)$, we get for all $n \geq 1$

$$\begin{aligned}\int_0^{2\pi} \varphi(x)\sin(nx)dx &= \left[-\varphi(x)\frac{\cos(nx)}{n}\right]_0^{2\pi} + \frac{1}{n}\int_0^{2\pi}\varphi'(x)\cos(nx)dx \\ &= \frac{1}{n}\int_0^{2\pi}\varphi'(x)\cos(nx)dx.\end{aligned}$$

Therefore, we get

$$\left|\int_0^{2\pi}\varphi(x)\sin(nx)dx\right| \leq \frac{1}{n}\int_0^{2\pi}|\varphi'(x)|dx \underset{n\to\infty}{\longrightarrow}.$$

The general case follows by density of $C_c^\infty(]0, 2\pi[)$ in $L^p(]0, 2\pi[)$ for $1 \leq p < \infty$.

Exercise 3. 1. Let $\varphi \in L^\infty(\Omega)$. We have

$$\int_\Omega (u_n v_n - u_\infty v_\infty)\,\varphi\,dx = \int_\Omega u_n\,(v_n - v_\infty)\,\varphi\,dx + \int_\Omega (u_n - u_\infty)\,v_\infty\,\varphi\,dx.$$

As $u_n \underset{n\to\infty}{\rightharpoonup} u_\infty$, the sequence $\left\{\|u_n\|_{\mathrm{L}^p(\Omega)}\right\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ is bounded, i.e., there exists $C < \infty$ such that $\|u_n\|_{\mathrm{L}^p(\Omega)} \leq C$ for all $n \in \mathbb{N}$. Therefore, Hölder's inequality implies that

$$\left|\int_\Omega u_n\,(v_n - v_\infty)\,\varphi\,dx\right| \leq \|\varphi\|_{\mathrm{L}^\infty(\Omega)}\|u_n\|_{\mathrm{L}^p(\Omega)}\|v_n - v\|_{\mathrm{L}^{p'}(\Omega)} \leq C\,\|\varphi\|_{\mathrm{L}^\infty(\Omega)}\|v_n - v_\infty\|_{\mathrm{L}^{p'}(\Omega)} \underset{n\to\infty}{\longrightarrow} 0.$$

Since $u_n - u_\infty \underset{n\to\infty}{\rightharpoonup} 0$, we deduce as $v_\infty\,\varphi \in L^{p'}(\Omega)$ that

$$\lim_{n\to\infty}\int_\Omega (u_n - u_\infty)\,v_\infty\,\varphi\,dx = 0$$

and the proof is complete. For the counter-example, take $u_n(x) = v_n(x) = \sin(nx)$ and use the previous exercise (or the Riemann-Lebesgue lemma).

2. We have

$$\int_\Omega |u_n - u_\infty|^2 dx = \int_\Omega u_n^2 dx - 2\int_\Omega u_n u_\infty dx + \int_\Omega u^2.$$

By assumption, we have

$$\int_\Omega u_n^2 dx \underset{n\to\infty}{\longrightarrow} \int_\Omega u^2 dx.$$

On the other hand, we get by the weak convergence

$$\int_\Omega u_n u_\infty dx \underset{n\to\infty}{\longrightarrow} \int_\Omega u^2 dx$$

and the proof is complete.

Exercise 4. 1. Thanks to Banach-Alaoglu-Bourbaki theorem, there exists $\overline{f} \in L^p(\Omega)$ such that $f_n \underset{n\to\infty}{\rightharpoonup} \overline{f}$ weakly. Then, since both limits coincide, we deduce that $f = \overline{f}$ and the proof is complete.

2. By Egorov's theorem, for all $\varepsilon > 0$, there exists $A \subset \Omega$ such that $|A| < \varepsilon$ and $f_n \underset{n\to\infty}{\longrightarrow} f$ uniformly in $\Omega \setminus A$. Write

$$\begin{aligned}\int_\Omega |f_n - f|^q dx &= \int_{\Omega\setminus A} |f_n - f|^q dx + \int_A |f_n - f|^q dx \\ &\leq \|f_n - f\|^q_{\mathrm{L}^\infty(\Omega\setminus A)} |\Omega| + \|f_n - f\|^q_{\mathrm{L}^p(\Omega)} |A|^{1-\frac{q}{p}} \\ &\leq \|f_n - f\|^q_{\mathrm{L}^\infty(\Omega\setminus A)} |\Omega| + C^q \varepsilon^{1-\frac{q}{p}},\end{aligned}$$

where we used that $\left\{\|f_n\|_{\mathrm{L}^p(\Omega)}\right\}_{n\in\mathbb{N}}$ is bounded due to the weak convergence.

Remarquek 1. One can also proceed by truncating the function.

Exercise 5. 1. By homogeneity, it suffices to show that

$$\sup_{-1\leq t\leq 1} \frac{\left||t+1|^p - |t|^p - 1\right|}{|t|^{p-1} + |t|} < \infty,$$

but this holds trivially.

2. We have

$$\left||f_n|^p - |f_n - f|^p - |f|^p\right| \leq C\left(|f_n - f|^p|f| + |f_n - f||f|^{p-1}\right).$$

By the previous exercise, we have $|f_n - f| \underset{n\to\infty}{\rightharpoonup} 0$ weakly in L^p and $|f_n - f|^{p-1} \underset{n\to\infty}{\rightharpoonup} 0$ weakly in $L^{p'}$, which shows that

$$\int_\Omega \Big||f_n|^p - |f_n - f|^p - |f|^p\Big| dx \leq C\int_\Omega \left(|f_n - f|^p|f| + |f_n - f||f|^{p-1}\right) dx \underset{n\to\infty}{\longrightarrow} 0.$$

The identity follows.

Indeed, the previous question shows that

$$\int_\Omega |f_n - f|^p dx = -\int_\Omega |f_n|^p dx + \int_\Omega \left(|f_n|^p - |f_n - f|^p\right) dx \underset{n\to\infty}{\longrightarrow} -\|f\|^p_{\mathrm{L}^p(\Omega)} + \int_\Omega |f|^p dx = 0.$$